Analyzing shape, accuracy, and precison of shooting results with `shotGroups`

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1 Introduction

The `shotGroups` package adds functionality to the open source statistical environment R (R Development Core Team, 2025a).[1] It provides functions to read in, plot, statistically describe, analyze, and compare shooting data with respect to group shape, precision, and accuracy.

- The functionality includes graphical methods, descriptive statistics, and inference tests using standard, but also non-parametric and robust statistical techniques (section 2).
- The package includes limited support for the analysis of three-dimensional data (sections 3.2.1, 3.2.2).
- Inference from range statistics like extreme spread is also supported (section 3.4).
- The data can be imported from files produced by OnTarget PC and OnTarget TDS (Block, 2016), Silver Mountain e-target (Silver Mountain Targets, 2018), ShotMarker e-target (MacDonald, 2018), SIUS e-target (SIUS AG, 2023), Taran (Trofimov, 2016), or from custom data files in text format with a similar structure.

Use `help(package="shotGroups")` for a list of all functions and links to the detailed help pages with information on options, usage and output.

For users who are unfamiliar with R, `shotGroups` includes a set of `shiny`-based web applications (Chang, Cheng, Allaire, Xie, & McPherson, 2024) running locally that eliminate the need to use R syntax. The applications implement different aspects of the functionality of `shotGroups`:

- `runGUI("analyze")` – Comprehensive shot group analysis and visualization (sections 2.3, 2.4, 2.5, 3.5)[2]
- `runGUI("hitprob")` – Region ↔ hit probability calculations (sections 3.2.2, 3.2.2)[3]
- `runGUI("range")` – Estimate Rayleigh σ from range statistics and do efficiency calculations for group statistics (section 3.4)[4]
- `runGUI("angular")` – Absolute ↔ angular size conversions (section 3.7)[5]

2 Analyzing bullet hole data

Analyzing shot groups usually takes the following steps:

- Read in data (section 2.1)
- Perform either a comprehensive numerical as well as graphical analysis of a group's shape, spread (precision), and location (accuracy) with `analyzeGroup()` (section 2.2) ...
- ...or analyze these aspects of a group separately with `groupShape()` (section 2.3), `groupSpread()` (section 2.4), `groupLocation()` (section 2.5)

[1] For an introduction to R, see Dalgaard (2008) or Quick-R (`https://www.statmethods.net/`).
[2] `http://shiny.imbei.uni-mainz.de:3838/shotGroups_AnalyzeGroups/`
[3] `http://shiny.imbei.uni-mainz.de:3838/shotGroups_HitProb/`
[4] `http://shiny.imbei.uni-mainz.de:3838/shotGroups_RangeStat/`
[5] `http://shiny.imbei.uni-mainz.de:3838/shotGroups_AngularSize/`

- Numerically and visually compare different groups in terms of their shape, location (accuracy), and spread (precision) with `compareGroups()` (section 2.6)
- Use additional utility functions to individually explore different aspects of a given group (section 3)

Grubbs (1964b) and `http://ballistipedia.com/` are good sources for statistical methods for analyzing shot groups.

2.1 Reading in data

To import data into R, it should be saved as a text file with the following format:

- The file should have one row for each shot, and one column for each coordinate as well as for any other variable such as distance to target, point-of-aim coordinates.
- Columns should be separated by commas, tabs or other whitespace. This type of text file can be exported from OnTarget PC/TDS, Silver Mountain e-target, ShotMarker e-target, SIUS e-target, Taran, or from a spreadsheet application like Excel or Calc.
- The file needs a header in the first line giving the variable names, and should contain at least the coordinates of points of impact, either with variable names `Point.X`, `Point.Y` or just `X`, `Y`.
- For several analysis functions, the following additional variables are useful: `Group` (group number), `Distance` (distance to target), and `Aim.X`, `Aim.Y` (point of aim). If these variables are missing, default values are assumed with a warning.
- If you have output files from OnTarget PC/TDS, you can read multiple files with `readDataOT1()` (for OnTarget PC v1.*, tested with v1.10), or with `readDataOT2()` (for OnTarget PC v2.* - tested with v2.28, and OnTarget TDS - tested with v3.71, v3.89, v6.09).
- If you have output files from the Silver Mountain e-target system, you can read multiple files with `readDataSMT()`.
- If you have CSV output files or .tar backup files from the ShotMarker e-target system, you can read multiple files with `readDataShotMarker()`.
- If you have CSV output files from the SIUS e-target system, you can read multiple files with `readDataSIUS()`.
- If you have other whitespace or comma-separated text files with the structure outlined above (e. g. from Taran), you can read multiple files with `readDataMisc()`. For three-dimensional data, this function also recognizes variables `Point.Z` or `Z` and `Aim.Z`.
- If your data is saved in some other text file format, consult the help for `read.table()` or the R import/export manual (R Development Core Team, 2025b).

```
library(shotGroups, verbose=FALSE)       # load shotGroups package

## read text files and save to data frame
## not run, we later use data frame provided in package instead
DFgroups <- readDataMisc(fPath="c:/path/to/files",
                         fNames=c("series1.dat", "series2.dat"))
```

By default, OnTarget's "Export Point Data" places the origin of the coordinate system in the top-left corner. This can be taken into account by correctly setting the option `xyTopLeft` in functions `analyzeGroup()` (section 2.2), `compareGroups` (section 2.6), and `drawGroup()` (section 3.5). In OnTarget TDS, the orientation of the y-axis can be changed by checking the box "Tools → Options → Options tab → Data Export → Invert Y-Axis on Export". If groups appear to be upside-down, `xyTopLeft` is the setting to change.

When analyzing different aspects of a group separately using `groupShape()` (section 2.3), `groupSpread()` (section 2.4), and `groupLocation()` (section 2.5), the scatterplots will be upside-down if the default option of OnTarget was used.

2.2 Performing a combined analysis

`analyzeGroup()`: This function is a convenience wrapper for the functions presented in sections 2.3, 2.4, and 2.5. It analyzes a group's shape, precision, and accuracy in one go, and collects the results.

```
library(shotGroups, verbose=FALSE)       # load shotGroups package
analyzeGroup(DFtalon, dstTarget=10, conversion="m2mm")

## output not shown, see following sections for results
```

2.3 Analyzing group shape

`groupShape()`: Assess (multivariate) normality, identify outliers and get a sense for the shape of the bivariate distribution.

Reported statistical parameters and tests:

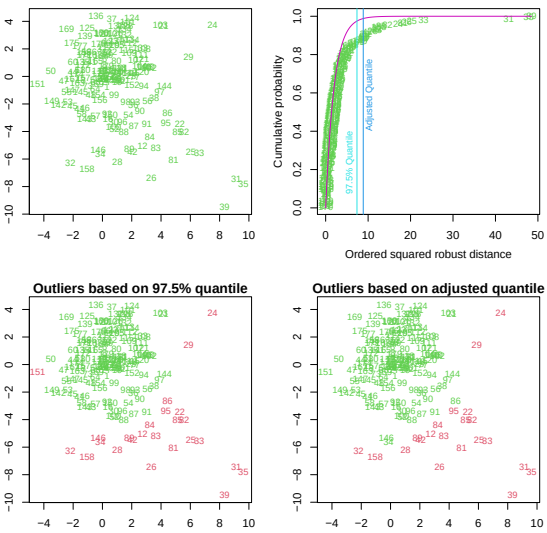
- Correlation matrix including a robust estimate using the MCD method (from package `robustbase`; Rousseeuw et al., 2024).
- Outlier identification – requires installing package `mvoutlier` (Filzmoser & Gschwandtner, 2021) first: Either using squared robust Mahalanobis distances and adjusted quantiles from the χ^2-distribution, or using robust principal components analysis (PCA) with options to tune the sensitivity.
- Shapiro-Wilk normality tests for the distribution of x- and y-coordinates. For more than 5000 observations, the drop-in Kolmogorov-Smirnov-test is reported instead.
- Energy test for bivariate normality of (x, y)-coordinates – requires installing package `energy` (Rizzo & Szekely, 2022) first.

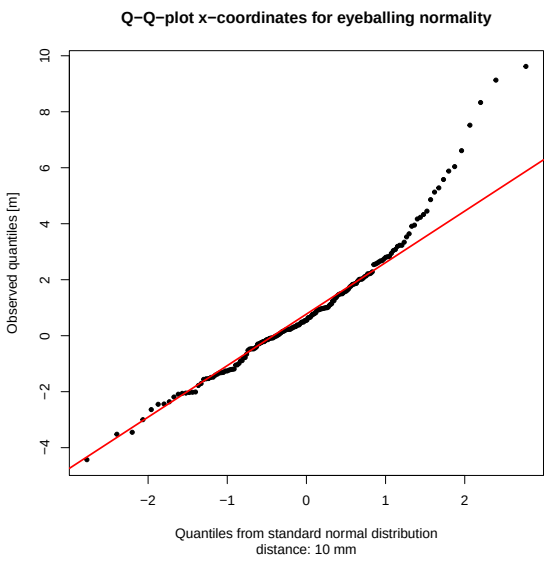
Plots:

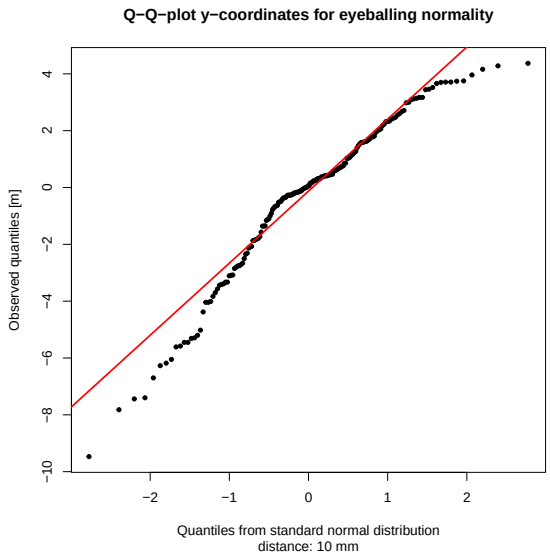
- Combined plot for multivariate outlier identification using squared robust Mahalanobis distances and adjusted quantiles from the χ^2-distribution – requires installing `mvoutlier`
- χ^2 QQ-plot of squared robust Mahalanobis distances to group center for eyeballing multivariate normality of (x, y)-coordinates
- Heatmap of a non-parametric 2D-kernel density estimate for the (x, y)-coordinates (from package `KernSmooth`; Wand, 2025) together with robust group center and robust error ellipse

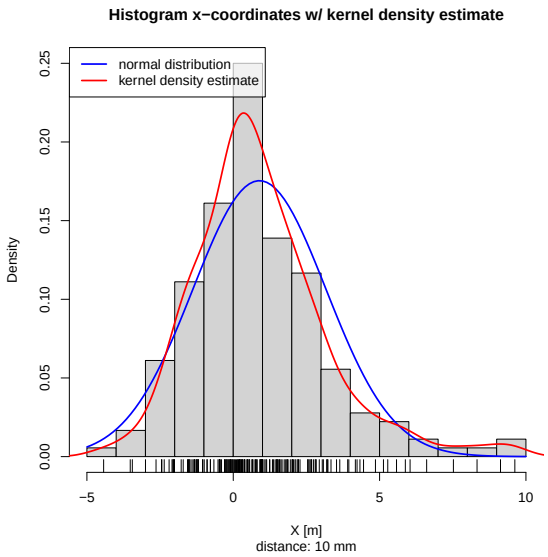
- QQ-plots of x- and y-coordinates for eyeballing normality
- Histogram of x- and y-coordinates including a fitted normal distribution as well as a non-parametric kernel density estimate

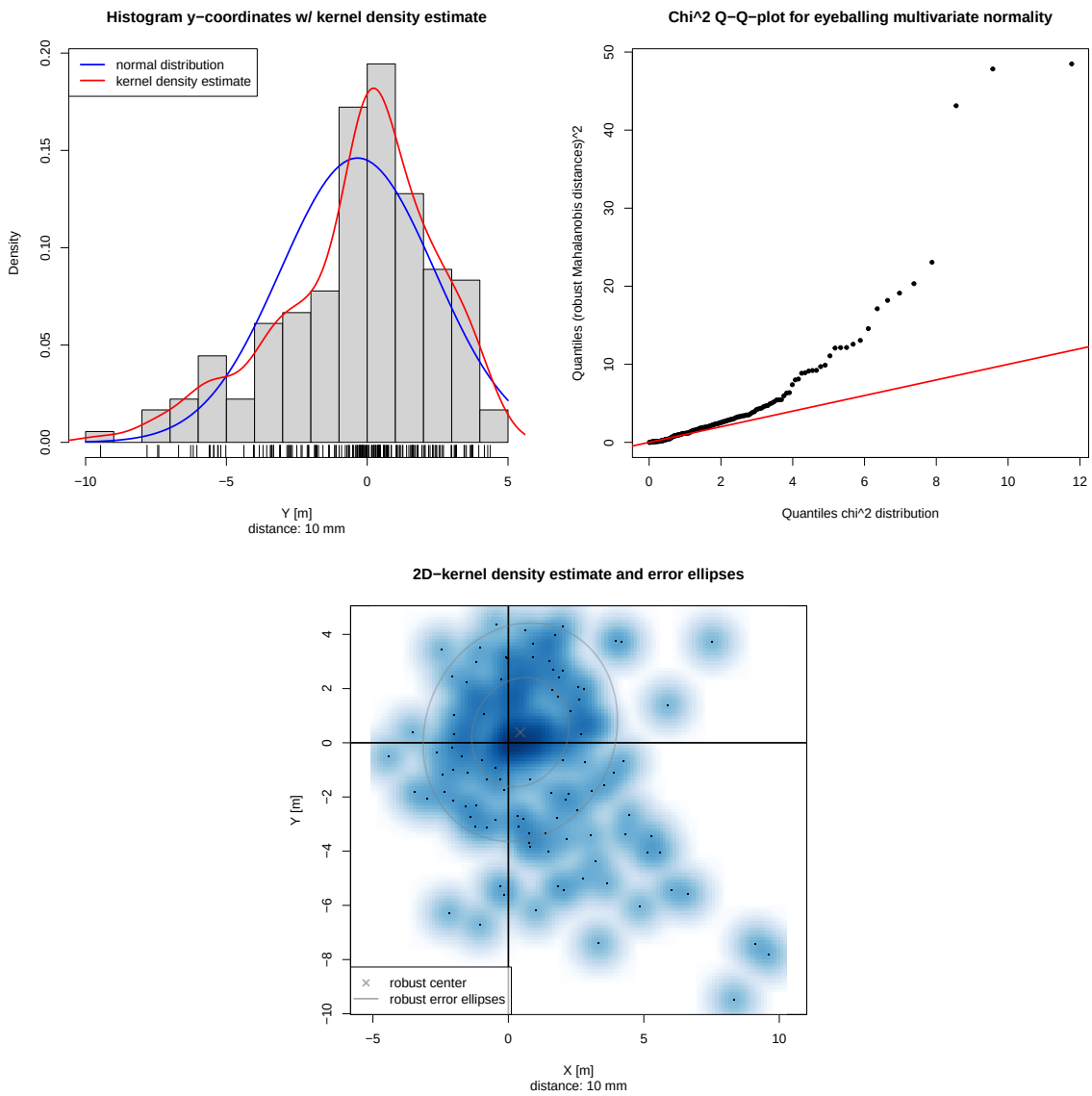
```
library(shotGroups, verbose=FALSE)       # load shotGroups package
groupShape(DFtalon, bandW=0.4, outlier="mcd",
           dstTarget=10, conversion="mm2m")

$corXY
        x       y
x  1.0000 -0.2931
y -0.2931  1.0000

$corXYrob
       x      y
x 1.0000 0.1083
y 0.1083 1.0000

$Outliers
 [1]  12  22  24  25  26  28  29  31  32  33  35  39  42  81  82  83  84
[18]  85  89  95 158

$ShapiroX

 Shapiro-Wilk normality test

data:  X
W = 0.95, p-value = 3e-06


$ShapiroY

 Shapiro-Wilk normality test

data:  Y
W = 0.96, p-value = 2e-05


$multNorm

 Energy test of multivariate normality: estimated parameters

data:  x, sample size 180, dimension 2, replicates 1499
E-statistic = 3.7, p-value <2e-16
```


Cumulative probability
Ordered squared robust distance
97.5% Quantile
Adjusted Quantile
Outliers based on 97.5% quantile
Outliers based on adjusted quantile


Q−Q−plot x−coordinates for eyeballing normality
Observed quantiles [m]
Quantiles from standard normal distribution
distance: 10 mm


Q−Q−plot y−coordinates for eyeballing normality
Observed quantiles [m]
Quantiles from standard normal distribution
distance: 10 mm


Histogram x−coordinates w/ kernel density estimate
normal distribution
kernel density estimate
Density
X [m]
distance: 10 mm



2.4 Analyzing group spread – precision

`groupSpread()`: Assess precision using empirical and parametric spread measures with confidence intervals. Where possible, also use the MCD method for a robust estimate of the covariance matrix (from package `robustbase`). Bootstrap confidence intervals are from package `boot` (Canty & Ripley, 2024) with 1499 replications.

Reported statistical parameters and tests:

- (Robust) Standard deviations of x- and y-coordinates together with parametric and bootstrap confidence intervals (in original measurement units, MOA, SMOA, mrad)
- (Robust) Covariance matrix of (x, y)-coordinates
- Empirical mean and median radius as well as estimated Rayleigh precision parameter σ, estimated Rayleigh radial standard deviation $\text{RSD} = \sigma\sqrt{\frac{4-\pi}{2}}$, and estimated Rayleigh

mean radius MR $= \sigma\sqrt{\frac{\pi}{2}}$ together with parametric and bootstrap confidence intervals for σ, RSD, and MR (in original measurement units, MOA, SMOA, mrad)

- Maximum pairwise distance (center-to-center, = maximum spread, in original measurement units, MOA, SMOA, mrad)
- Width and height of bounding box with length of diagonal and figure of merit as well as of the (oriented) minimum-area bounding box (in original measurement units, MOA, SMOA, mrad)
- Radius for the minimum enclosing circle (in original measurement units, MOA, SMOA, mrad)
- Length of semi-major and semi-minor axis of the (robust) confidence ellipse (in original measurement units, MOA, SMOA, mrad)
- Aspect ratio $\sqrt{\kappa}$ (with condition index κ) and flattening $1 - \frac{1}{\sqrt{\kappa}}$ of the (robust) confidence ellipse as well as the trace and determinant of the covariance matrix
- Estimate for the circular error probable CEP (section 3.2.1; in original measurement units, MOA, SMOA, mrad)

Plots:

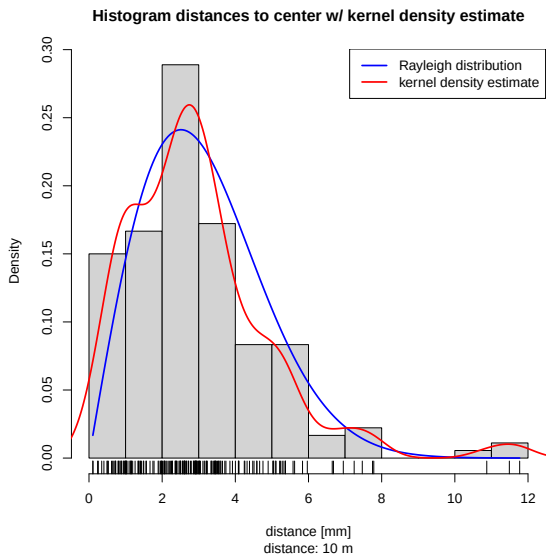
- Scatterplot of the (x, y)-coordinates together with group center, circle with average distance to center, and (robust) confidence ellipse
- Scatterplot of the (x, y)-coordinates together with the bounding box, minimum-area bounding box, minimum enclosing circle, and maximum group spread
- Histogram of distances to group center including a Rayleigh fit and a non-parametric kernel density estimate

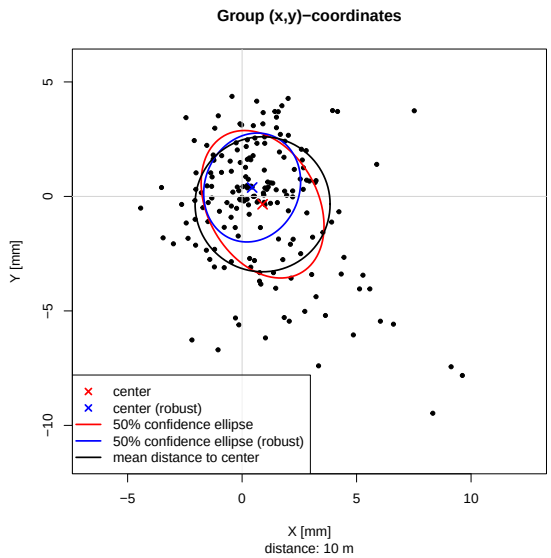
```
library(shotGroups, verbose=FALSE)       # load shotGroups package
groupSpread(DFtalon, CEPtype=c("CorrNormal", "GrubbsPatnaik", "Rayleigh"),
            CEPlevel=0.5, CIlevel=0.95, bootCI="basic",
            dstTarget=10, conversion="m2mm")

$sdXY
          x      y
unit 2.2746 2.7308
MOA  0.7819 0.9388
SMOA 0.8188 0.9831
mrad 0.2275 0.2731

$sdXci
      sdX (  sdX ) sdX basic ( sdX basic )
unit 2.0614 2.5374      1.9479      2.6076
MOA  0.7087 0.8723      0.6696      0.8964
SMOA 0.7421 0.9134      0.7012      0.9387
mrad 0.2061 0.2537      0.1948      0.2608

$sdYci
```

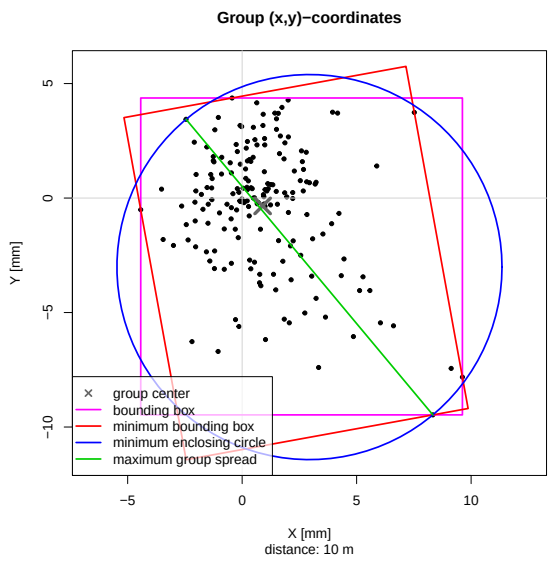
```
      sdY (  sdY ) sdY basic ( sdY basic )
unit 2.4749 3.0463      2.4384       3.060
MOA  0.8508 1.0472      0.8383       1.052
SMOA 0.8909 1.0967      0.8778       1.102
mrad 0.2475 0.3046      0.2438       0.306

$sdXYrob
          x      y
unit 1.7945 2.0160
MOA  0.6169 0.6931
SMOA 0.6460 0.7258
mrad 0.1794 0.2016

$covXY
       x      y
x  5.174 -1.820
y -1.820  7.457

$covXYrob
       x      y
x 3.2201 0.3918
y 0.3918 4.0644

$distToCtr
       mean median    max  sigma    RSD     MR
unit 2.9486 2.6696 11.772 2.5148 1.6476 3.1519
MOA  1.0137 0.9178  4.047 0.8645 0.5664 1.0835
SMOA 1.0615 0.9611  4.238 0.9053 0.5931 1.1347
mrad 0.2949 0.2670  1.177 0.2515 0.1648 0.3152

$sigmaCI
     sigma ( sigma ) sigma basic ( sigma basic )
unit  2.3433  2.7136        2.2419        2.8032
MOA   0.8056  0.9329        0.7707        0.9637
SMOA  0.8436  0.9769        0.8071        1.0091
mrad  0.2343  0.2714        0.2242        0.2803

$RSDci
      RSD (  RSD ) RSD basic ( RSD basic )
unit 1.5352 1.7778      1.4687      1.8365
MOA  0.5278 0.6112      0.5049      0.6313
SMOA 0.5527 0.6400      0.5287      0.6611
mrad 0.1535 0.1778      0.1469      0.1836

$MRci
       MR (   MR ) MR basic ( MR basic )
```

```
unit 2.9369 3.4010     2.8098     3.5133
MOA  1.0096 1.1692     0.9659     1.2078
SMOA 1.0573 1.2244     1.0115     1.2648
mrad 0.2937 0.3401     0.2810     0.3513

$maxPairDist
  unit    MOA   SMOA   mrad
16.819  5.782  6.055  1.682

$groupRect
      width height    FoM   diag
unit 14.050 13.840 13.945 19.722
MOA   4.830  4.758  4.794  6.780
SMOA  5.058  4.982  5.020  7.100
mrad  1.405  1.384  1.394  1.972

$groupRectMin
      width height    FoM   diag
unit 15.185 12.517 13.851 19.679
MOA   5.220  4.303  4.762  6.765
SMOA  5.466  4.506  4.986  7.084
mrad  1.518  1.252  1.385  1.968

$minCircleRad
  unit    MOA   SMOA   mrad
8.4095 2.8910 3.0274 0.8409

$minEll
     semi_major semi_minor
unit     8.7976     7.0522
MOA      3.0244     2.4244
SMOA     3.1671     2.5388
mrad     0.8798     0.7052

$confEll
     semi-major semi-minor
unit     3.4321     2.4081
MOA      1.1799     0.8278
SMOA     1.2356     0.8669
mrad     0.3432     0.2408

$confEllRob
     semi-major semi-minor
unit     2.4441     2.0735
MOA      0.8402     0.7128
SMOA     0.8799     0.7464
```

```
mrad      0.2444      0.2073

$confEllShape
      angle aspectRatio  flattening       trace         det
   118.9509      1.4253      0.2984     12.6311     35.2687

$confEllShapeRob
      angle aspectRatio  flattening       trace         det
    71.7748      1.1787      0.1516      7.3816     13.2601

$CEP
$CEP$CEP0.5
     CorrNormal GrubbsPatnaik Rayleigh
unit     2.9013        2.8913   2.9610
MOA      0.9974        0.9939   1.0179
SMOA     1.0445        1.0409   1.0660
mrad     0.2901        0.2891   0.2961
```







2.5 Analyzing group location – accuracy

`groupLocation()`: Assess accuracy of a group using empirical and parametric measures. Where possible, also use the MCD method for a robust estimate of the covariance matrix (from package `robustbase`). Bootstrap confidence intervals are from package `boot` with 1499 replications.

Reported statistical parameters and tests:

- (x, y)-offset of (robust) group center relative to point of aim
- Distance from (robust) group center to point of aim (in original measurement units, MOA, SMOA, mrad)
- Hotelling's T^2-test result for equality of the true group center with point of aim
- Parametric and bootstrap confidence intervals for the true center's x- and y-coordinate

Plots:

- Scatterplot of the (x, y)-coordinates together with (robust) group center.

```
library(shotGroups, verbose=FALSE)       # load shotGroups package
groupLocation(DFtalon, dstTarget=10, conversion="m2mm",
              level=0.95, plots=FALSE, bootCI="basic")

$ctr
      x       y
 0.8947 -0.3432

$ctrRob
     x      y
0.4646 0.3696

$distPOA
```

```
   unit     MOA    SMOA    mrad
0.95828 0.32943 0.34498 0.09583

$distPOArob
   unit     MOA    SMOA    mrad
0.59373 0.20411 0.21374 0.05937

$Hotelling
Analysis of Variance Table

             Df Hotelling-Lawley approx F num Df den Df  Pr(>F)
(Intercept)   1            0.156     13.9      2    178 2.5e-06
Residuals   179

$ctrXci
         x (   x )
t     0.5602 1.229
basic 0.5633 1.232

$ctrYci
          y (     y )
t     -0.7448 0.05849
basic -0.7211 0.07410
```

2.6 Comparing groups

`compareGroups()`: Compare two or more groups with regard to their precision and accuracy using empirical measures and statistical tests.

`compareGroups()` requires that the data includes a variable `series` that identifies shot groups. OnTarget PC/TDS' variable `group` identifies groups just within one file, `series` should number groups also across different original files. When you read in data with `readDataOT1()`, `series` is added automatically (same for `readDataOT2()`, `readDataSMT()`, `readDataShotMarker()`, `readDataSIUS()`, and `readDataMisc()`). For data from just one file, you can otherwise copy variable `group` to `series` in a data frame called `shots` with

```
shots$series <- shots$group
```

Reported statistical parameters and tests:

- Group center offset from the respective point of aim
- Distances from group centers to their respective point of aim (in original measurement units, MOA, SMOA, mrad)
- MANOVA result from testing equality of group center offset from the respective point of aim
- Group correlation matrices for the (x, y)-coordinates

- Group standard deviations of the x- and y-coordinates including parametric 95%-confidence intervals (in original measurement units, MOA, SMOA, mrad)
- Average distances from points to their respective group center (in original measurement units, MOA, SMOA, mrad)
- Maximum pairwise distance between points for each group (center-to-center, = maximum spread, in original measurement units, MOA, SMOA, mrad)
- Figure of merit FoM and diagonal of the (oriented) minimum-area bounding box for each group (in original measurement units, MOA, SMOA, mrad)
- Radius of the minimum enclosing circle for each group (in original measurement units, MOA, SMOA, mrad)
- Estimated Rayleigh parameter σ (precision) for each group (in original measurement units, MOA, SMOA, mrad)
- Estimated Rayleigh mean radius MR for each group (in original measurement units, MOA, SMOA, mrad)
- Parametric χ^2 confidence intervals for Rayleigh σ and MR (in original measurement units, MOA, SMOA, mrad)
- Estimate for the 50% circular error probable (CEP) in each group (section 3.2.1; in original measurement units, MOA, SMOA, mrad)
- Ansari-Bradley-test results from testing equality of group variances for x- and y-coordinates – when two groups are compared. With more than two groups, the Fligner-Killeen-test is used
- Wilcoxon-Rank-Sum-test (= Mann-Whitney-U-test) result from testing equality of average point distances to their respective group center – when two groups are compared. With more than two groups, the Kruskal-Wallis-test is used

The Ansari-Bradley-, Fligner-Killeen-, Wilcoxon-Rank-Sum-, and Kruskal-Wallis-tests are implemented as permutation tests using the `coin` package (Hothorn, Hornik, van de Wiel, & Zeileis, 2008). The tests for two groups (Ansari-Bradley, Wilcoxon) use the exact permutation distribution, the tests for more than two groups (Fligner-Killeen, Kruskal-Wallis) use the approximate permutation distribution with 9999 random permutations.

Plots:

- Scatterplot showing all groups as well as their respective center and 50%-confidence ellipse
- Scatterplot showing all groups as well as their respective (minimum) bounding box and maximum group spread
- Scatterplot showing all groups as well as their respective minimum enclosing circle and circle with average distance to center
- Boxplot for the distances to group center per group
- Stripchart showing the distances to group center per group together with the estimated Rayleigh mean radius and its confidence interval

```
library(shotGroups, verbose=FALSE)       # load shotGroups package

## only use first 3 groups of DFtalon
```

```r
DFsub <- subset(DFtalon, series %in% 1:3)
compareGroups(DFsub, dstTarget=10, conversion="m2mm")
```

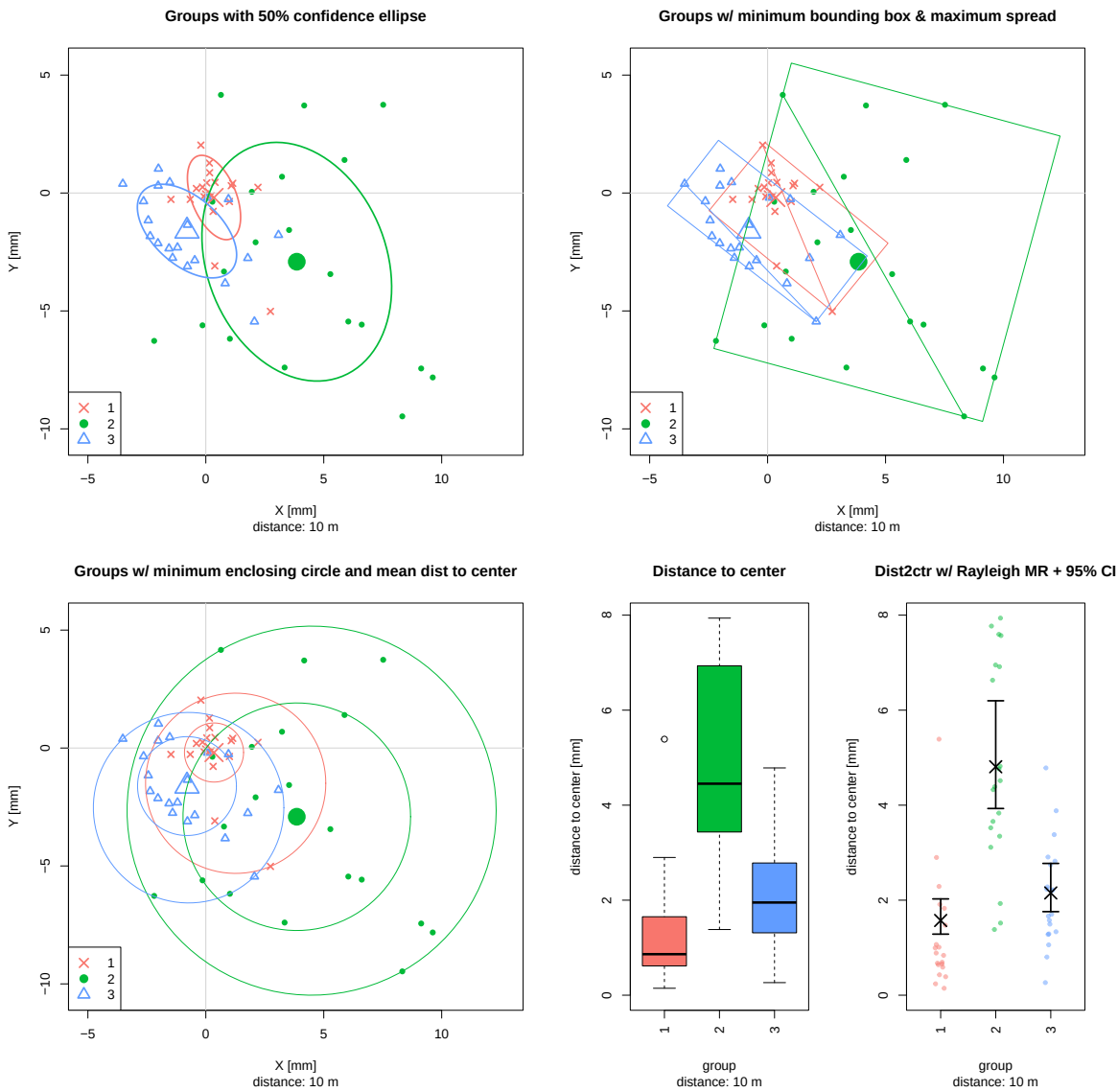
```
$ctr
        1      2       3
x  0.3475  3.856 -0.7985
y -0.1910 -2.913 -1.6140

$distPOA
           1      2      3
unit 0.39653 4.8329 1.8007
MOA  0.13632 1.6614 0.6190
SMOA 0.14275 1.7399 0.6483
mrad 0.03965 0.4833 0.1801

$MANOVA
Analysis of Variance Table

            Df Wilks approx F num Df den Df  Pr(>F)
(Intercept)  1 0.676     13.4      2     56 1.7e-05
series       2 0.504     11.4      4    112 7.9e-08
Residuals   57

$corXY
$corXY$`1`
        x       y
x  1.0000 -0.4632
y -0.4632  1.0000

$corXY$`2`
        x       y
x  1.0000 -0.2143
y -0.2143  1.0000

$corXY$`3`
        x       y
x  1.0000 -0.5081
y -0.5081  1.0000


$sdXY
$sdXY$`1`
           x      y
unit 0.94037 1.4912
MOA  0.32327 0.5126
SMOA 0.33853 0.5368
```

```
mrad 0.09404 0.1491

$sdXY$`2`
          x     y
unit 3.3539 4.220
MOA  1.1530 1.451
SMOA 1.2074 1.519
mrad 0.3354 0.422

$sdXY$`3`
          x      y
unit 1.7549 1.6556
MOA  0.6033 0.5691
SMOA 0.6318 0.5960
mrad 0.1755 0.1656


$sdXYci
$sdXYci$`1`
       sdX (  sdX )  sdY (  sdY )
unit 0.71514 1.3735 1.1340 2.1780
MOA  0.24585 0.4722 0.3898 0.7487
SMOA 0.25745 0.4945 0.4082 0.7841
mrad 0.07151 0.1373 0.1134 0.2178

$sdXYci$`2`
      sdX (  sdX )  sdY (  sdY )
unit 2.5506 4.8986 3.2094 6.1638
MOA  0.8768 1.6840 1.1033 2.1190
SMOA 0.9182 1.7635 1.1554 2.2190
mrad 0.2551 0.4899 0.3209 0.6164

$sdXYci$`3`
      sdX (  sdX )  sdY (  sdY )
unit 1.3346 2.5631 1.2591 2.4181
MOA  0.4588 0.8811 0.4328 0.8313
SMOA 0.4804 0.9227 0.4533 0.8705
mrad 0.1335 0.2563 0.1259 0.2418


$meanDistToCtr
          1      2      3
unit 1.2526 4.8246 2.0961
MOA  0.4306 1.6586 0.7206
SMOA 0.4509 1.7368 0.7546
mrad 0.1253 0.4825 0.2096
```

```
$maxPairDist
          1      2      3
unit 7.6423 15.650 8.0773
MOA  2.6272  5.380 2.7768
SMOA 2.7512  5.634 2.9078
mrad 0.7642  1.565 0.8077

$bbFoM
          1      2      3
unit 5.2415 12.170 5.7565
MOA  1.8019  4.184 1.9790
SMOA 1.8869  4.381 2.0724
mrad 0.5241  1.217 0.5757

$bbDiag
          1      2      3
unit 7.7121 17.219 8.7324
MOA  2.6512  5.920 3.0020
SMOA 2.7764  6.199 3.1437
mrad 0.7712  1.722 0.8732

$minCircleRad
          1      2      3
unit 3.8212 7.8248 4.0386
MOA  1.3136 2.6900 1.3884
SMOA 1.3756 2.8169 1.4539
mrad 0.3821 0.7825 0.4039

$sigma
          1      2      3
unit 1.2548 3.8369 1.7172
MOA  0.4314 1.3190 0.5903
SMOA 0.4517 1.3813 0.6182
mrad 0.1255 0.3837 0.1717

$MR
          1      2      3
unit 1.5727 4.8088 2.1522
MOA  0.5406 1.6531 0.7399
SMOA 0.5662 1.7312 0.7748
mrad 0.1573 0.4809 0.2152

$sigmaMRci
$sigmaMRci$`1`
     sigma ( sigma )   MR (   MR )
unit  1.0255  1.6172 1.2852 2.0268
```

```
MOA   0.3525  0.5559 0.4418 0.6968
SMOA  0.3692  0.5822 0.4627 0.7296
mrad  0.1025  0.1617 0.1285 0.2027

$sigmaMRci$`2`
     sigma ( sigma )  MR (   MR )
unit  3.1357  4.9449 3.930 6.1975
MOA   1.0780  1.6999 1.351 2.1305
SMOA  1.1288  1.7801 1.415 2.2311
mrad  0.3136  0.4945 0.393 0.6197

$sigmaMRci$`3`
     sigma ( sigma )   MR (   MR )
unit  1.4034  2.2131 1.7589 2.7737
MOA   0.4824  0.7608 0.6047 0.9535
SMOA  0.5052  0.7967 0.6332 0.9985
mrad  0.1403  0.2213 0.1759 0.2774


$CEP
          1      2      3
unit 1.3724 4.4169 1.9169
MOA  0.4718 1.5184 0.6590
SMOA 0.4941 1.5901 0.6901
mrad 0.1372 0.4417 0.1917

$FlignerX

 Approximative K-Sample Fligner-Killeen Test

data:  x by series (1, 2, 3)
chi-squared = 18, p-value = 1e-04


$FlignerY

 Approximative K-Sample Fligner-Killeen Test

data:  y by series (1, 2, 3)
chi-squared = 21, p-value <1e-04


$Kruskal

 Approximative Kruskal-Wallis Test
```

```
data:  dstCtr by series (1, 2, 3)
chi-squared = 30, p-value <1e-04
```



3 Additional functionality

The `shotGroups` package also provides a number of utility functions that can be used separately to ...

- calculate individual descriptive precision measures (section 3.1)
- estimate hit probabilities: either get the region that is expected to contain a certain fraction of shots, or get the estimated fraction of shots expected to be within a given region (section 3.2)

- estimate Rayleigh parameter σ from range statistics like extreme spread, and do efficiency calculations for several group statistics (section 3.4)
- plot a group to scale on a target background and add precision indicators (section 3.5)
- simulate the ring count for a given group, bullet diameter, and target type (section 3.6)
- convert between absolute and angular size units deg, MOA, SMOA, rad, and mrad (section 3.7)
- try an analysis on collections of empirical data included in the package (section 3.8)

3.1 Descriptive precision measures – range statistics

The following functions can be used to calculate so-called range statistics. These are precision measures that summarize a specific feature of the group's geometry that is related to the groups overall size. On average, range statistics grow with the number of shots per group, and therefore must be considered in relation to the number of shots they summarize. Section 3.4 discusses the distribution of range statistics, possibilities to use them for inference on true spread, and their statistical efficiency. Section 3.5 illustrates how to add these precision indicators to a plot of the group. Note that faster, and likely more numerically stable algorithms for bounding boxes / circles / ellipses exist, also for generalizing the problem to higher dimensions (Fischer, Gärtner, & Kutz, 2003; Computational Geometry Algorithms Library, 2021; Gärtner, 2021).

- `getBoundingBox()`: Calculates the vertices, length of diagonal, and figure of merit (FoM) of the axis-aligned bounding box. This is the smallest rectangle that contains all points (bullet hole centers), and has edges parallel to the x- and y-axis.
- `getMinBBox()`: Calculates the vertices, length of diagonal, and figure of merit (FoM) of the minimum-area bounding box. This is the smallest, possibly oriented rectangle that contains all points (bullet hole centers). Uses an approach similar to the rotating calipers algorithm (Toussaint, 1983).
- `getMinCircle()`: Calculates center and radius of the minimum enclosing circle. This is the smallest circle that contains all points (bullet hole centers). Uses the Skyum algorithm (Skyum, 1991).
- `getMinEllipse()`: Calculates center and shape matrix of the minimum enclosing ellipse. This is the smallest ellipse that contains all points (bullet hole centers). Uses Khachiyan's algorithm (Todd & Yildrim, 2007).
- `getMaxPairDist()`: Calculates the maximum of all pairwise distances between points – also called extreme spread, or group size.
- `getDistToCtr()`: Calculates the distances of a set of points to their center. The mean or median can then be taken as a precision measure. The mean distance to group center is not a range statistic as it includes information from all shots.

```
library(shotGroups, verbose=FALSE)        # load shotGroups package
getBoundingBox(DFtalon)                   # axis-aligned bounding box

$pts
  xleft ybottom  xright    ytop
  -4.43   -4.37    9.62    9.47
```

```
$width
[1] 14.05

$height
[1] 13.84

$FoM
[1] 13.95

$diag
[1] 19.72

getMinBBox(DFtalon)                        # minimum-area bounding box

$pts
          x      y
[1,] -2.447 11.428
[2,] -5.161 -3.512
[3,]  7.155 -5.750
[4,]  9.869  9.190

$width
[1] 15.18

$height
[1] 12.52

$FoM
[1] 13.85

$diag
[1] 19.68

$angle
   y
79.7

getMinCircle(DFtalon)                      # minimum enclosing circle

$ctr
[1] 2.940 3.015

$rad
[1] 8.409

getMinEllipse(DFtalon)                     # minimum enclosing ellipse
```

```
$ctr
[1] 3.726 2.242

$E
          [,1]      [,2]
[1,]  0.015815 -0.003525
[2,] -0.003525  0.017212

$cov
      [,1]  [,2]
[1,] 66.26 13.57
[2,] 13.57 60.88

$area
[1] 194.9

$shape
      angle aspectRatio  flattening       trace         det
    39.3945      1.2475      0.1984    127.1319   3849.3009

$size
[1] 8.798 7.052

getMaxPairDist(DFtalon)                  # extreme spread / group size

$d
[1] 16.82

$idx
[1] 169  39
```

3.2 Estimating hit probability

Beyond calculating decriptive/geometric precision measures, `shotGroups` also includes functions that provide inferential statistics to estimate hit probabilities.

- Section 3.2.1 shows how to estimate the circular, spherical or elliptical region that is expected to contain a given fraction of shots.
- Section 3.2.2 describes how to estimate the fraction of shots expected to be within a given distance to the true group center.
- Section 3.2.3 covers the extrapolation of hit probabilities to different distances other than the one a group was actually shot at.

3.2.1 Region for a given hit probability: CEP, SEP and confidence ellipse

The following functions estimate the region that is expected to contain a given fraction of shots (bullet hole centers) under different assumptions. The given fraction of shots is the same as the probability for one shot to lie within the calculated region. The functions can use the MCD method for a robust estimate of the group center and covariance matrix (from package `robustbase`).

- `getCEP()`: Calculates estimates for the Circular Error Probable CEP. For three-dimensional data, the Spherical Error Probable SEP is returned. The CEP/SEP estimate is the radius of the circle/sphere around the point of aim (POA) that is expected to cover a certain fraction of points. If systematic accuracy bias is ignored, the POA is assumed to coincide with the true group center. If systematic accuracy bias is taken into account, the POA is in the origin $(0, 0)$, possibly offset from the true group center. The following estimates are available:
  - `CorrNormal`: If systematic accuracy bias is ignored, and for two-dimensional data, this estimate is based on the Hoyt distribution for radial error in correlated bivariate normal variables re-written in polar coordinates (radius and angle; Hoyt, 1947; Paris, 2009a, 2009b). If systematic accuracy bias is taken into account, a numerical algorithm (Farebrother implemented in package `CompQuadForm`, Duchesne and Lafaye de Micheaux (2010)) is used to calculate the cumulative distribution function (cdf) of radial error from integration of the multivariate normal distribution over an offset disc (DiDonato & Jarnagin, 1961a; Evans, Govindarajulu, & Barthoulot, 1985) or sphere (DiDonato, 1988). The `CorrNormal` estimate is available for all probability levels.
    - `Krempasky`: The Krempasky (2003) estimate is based on a nearly exact closed-form solution for the 50% quantile of the Hoyt distribution. It is only available for `accuracy=FALSE` and does not generalize to three dimensions.
    - `Ignani`: The Ignani estimate (Ignani, 2010) is based on a polynomial approximation for the 50%, 90%, 95%, and 99% quantiles of the Hoyt distribution. It is only available for `accuracy=FALSE` and generalizes to three dimensions.
    - `RAND`: The modified RAND R-234 estimate (RAND Corporation, 1952) is based on lookup tables for the 50% quantile of the Hoyt distribution. The tables were later cast into an algebraic form that is essentially the Rayleigh estimator (see below) with a weighted average of the variances of the de-correlated data to estimate the true standard deviation. The bias correction with `accuracy=TRUE` is based on a cubic regression fit to tabulated data (Pesapane & Irvine, 1977; Puhek, 1992). This estimate does not generalize to three dimensions.
    - `Valstar`: The Valstar estimate (Puhek, 1992) for the 50% quantile of the Hoyt distribution differs from the RAND-estimate only for highly elliptical distributions and in its method of correcting for systematic accuracy bias. This estimate does not generalize to three dimensions.
  - `GrubbsPearson`: The Grubbs-Pearson estimate (Grubbs, 1964a) is based on the Pearson three-moment central χ^2-approximation (Imhof, 1961; Pearson, 1959) of the cdf of radial error in bivariate normal variables. Shot coordinates may be correlated and have unequal variances. The eigenvalues of the covariance matrix

of coordinates are used as variance estimates since they are the variances of the principal components (the PCA-rotated = decorrelated data). For probabilities ≥ 0.25, the approximation is very close to the cdf used in `CorrNormal` – but easier to calculate. For probabilities < 0.25 and some distribution shapes, the approximation can diverge from the actual cdf. The Grubbs-Pearson estimate is available for all probability levels, and generalizes to three dimensions.

  - `GrubbsPatnaik`: The Grubbs-Patnaik estimate (Grubbs, 1964a) differs from the Grubbs-Pearson estimate insofar as it is based on the Patnaik (1949) two-moment central χ^2-approximation of the true cdf of radial error. For probabilities < 0.5 and some distribution shapes, the approximation can diverge from the actual cdf.
  - `GrubbsLiu`: The Grubbs-Liu estimate was not proposed by Grubbs but follows the same principle as his original estimates. It differs from them insofar as it is based on the Liu, Tang, and Zhang (2009) four-moment non-central χ^2-approximation of the true cdf of radial error. For `accuracy=FALSE`, it is identical to `GrubbsPearson`.
  - `Rayleigh`: If systematic accuracy bias is ignored, and for two-dimensional data, this estimate uses the Rayleigh distribution (H. P. Singh, 1992). It is valid for uncorrelated bivariate normal coordinates with equal variances and zero mean. For `accuracy=FALSE` and three-dimensional data, the Maxwell-Boltzmann distribution is used. For `accuracy=TRUE` and two-dimensional data, the estimate uses the Rice distribution. For `accuracy=TRUE` and three-dimensional data, it is based on the offset sphere probabilities for the multivariate normal distribution set to have equal variances. This estimate is available for all probability levels.
  - `RMSE`: For `accuracy=FALSE`, this estimator is essentially the same as the RMSE estimator often described in the GPS literature (van Diggelen, 2007) when using centered data for calculating RMSE (square root of the mean squared error). It is very similar to the `Rayleigh` estimator. For `accuracy=TRUE`, this is essentially the same as the RMSE estimator often described in the GPS literature when using the original, non-centered data for calculating RMSE. It is similar to the `Rayleigh` estimator only when bias is small, but becomes seriously wrong otherwise. The `RMSE` estimate is available for all probability levels, and generalizes to three dimensions.
  - `Ethridge`: The Ethridge estimate (Ethridge, 1983) is not based on the assumption of multivariate normality of shot coordinates but uses a robust unbiased estimator for the median radius (Hogg, 1967). The Ethridge estimate is also documented in Puhek (1992).[6] This estimate can only be reported for probability 0.5 but generalizes to three dimensions.

- `getConfEll()`: Calculates the confidence ellipse for the true mean of the distribution under the assumption of multivariate normality of shot coordinates. The coordinates may be correlated and have unequal variances. The confidence ellipse gives the iso-probability contour, the points on its rim all have the same Mahalanobis distance to the center. The result also includes the ellipse based on a robust estimate for the covariance matrix of the coordinates using the MCD algorithm (from package `robustbase`). The confidence ellipse generalizes to three-dimensional data.

Estimates based on the normal distribution use the plug-in method (Blischke & Halpin, 1966),

[6] Note that the formula for the Hogg weighted location estimate is wrong in Puhek (1992); Tongue (1993); Wang, Yang, Jia, and Wang (2013); Wang, Yang, Yan, Wang, and Song (2014); Williams (1997).

i. e., they substitute the true covariance matrix and mean vector with those estimated from the data. They are thus strictly valid only for the asymptotic distribution, while the finite sample distribution may differ somewhat. See section 3.2.4 for more references to relevant literature, and section 3.3 for a discussion of the different distributions of radial error mentioned below.

```
## circular error probable
getCEP(DFscar17, type=c("GrubbsPatnaik", "Rayleigh"), CEPlevel=0.5,
       dstTarget=100, conversion="yd2in")

$CEP
$CEP$CEP0.5
     GrubbsPatnaik Rayleigh
unit        0.8415   0.8751
MOA         0.8036   0.8357
SMOA        0.8415   0.8751
mrad        0.2337   0.2431


$ellShape
aspectRatio  flattening
     1.4503      0.3105

$ctr
    x     y
2.599 2.299

## confidence ellipse
getConfEll(DFscar17, level=0.95,
           dstTarget=100, conversion="yd2in")

$ctr
    x     y
2.599 2.299

$ctrRob
    x     y
2.804 2.283

$cov
        x       y
x  0.4492 -0.1695
y -0.1695  0.6253

$covRob
         x        y
x 0.019656 0.004164
y 0.004164 1.055305
```

```
$size
     semi-major semi-minor
unit     2.4900     1.7168
MOA      2.3778     1.6395
SMOA     2.4900     1.7168
mrad     0.6917     0.4769

$sizeRob
     semi-major semi-minor
unit     2.9973     0.4089
MOA      2.8622     0.3905
SMOA     2.9973     0.4089
mrad     0.8326     0.1136

$shape
      angle aspectRatio  flattening       trace         det
   121.2738      1.4503      0.3105      1.0745      0.2522

$shapeRob
      angle aspectRatio  flattening       trace         det
   89.76961     7.33035     0.86358     1.07496     0.02073

$magFac
[1] 2.918
```

Function `getRayParam()` estimates the Rayleigh distribution's radial precision parameter σ together with its radial standard deviation $\mathrm{RSD} = \sigma\sqrt{\frac{4-\pi}{2}}$, and its mean radius $\mathrm{MR} = \sigma\sqrt{\frac{\pi}{2}}$, including parametric confidence intervals. For 3D data, it also estimates σ, $\mathrm{MR} = \sigma\sqrt{\frac{8}{\pi}}$ and $\mathrm{RSD} = \sigma\sqrt{\frac{3\pi-8}{\pi}}$ of the Maxwell-Boltzmann distribution.

```
## Rayleigh parameter estimates with 95% confidence interval
getRayParam(DFscar17, level=0.95)

$sigma
  sigma sigCIlo sigCIup
 0.7432  0.5616  1.0991

$RSD
    RSD RSDciLo RSDciUp
 0.4869  0.3679  0.7201

$MR
    MR MRciLo MRciUp
0.9315 0.7039 1.3775
```

```
$MEDR
    MEDR MEDRciLo MEDRciUP
  0.8751   0.6612   1.2941

## Maxwell-Boltzmann parameter estimates with 95% confidence interval
xyz <- matrix(rnorm(60), ncol=3)
getRayParam(xyz, level=0.95)

$sigma
  sigma sigCIlo sigCIup
 1.0992  0.9293  1.3458

$RSD
    RSD RSDciLo RSDciUp
 0.7402  0.6258  0.9063

$MR
    MR MRciLo MRciUp
 1.754  1.483  2.148

$MEDR
    MEDR MEDRciLo MEDRciUP
   1.691    1.429    2.070
```

3.2.2 Hit probability for a given region

Given a circle or sphere with radius r around the true mean of the bullet hole distribution, `getHitProb()` estimates the expected fraction of shots that has at most distance r to the group center. The estimated fraction of shots is the same as the estimated probability for one shot to lie in the circle with radius r. The probability can be calculated using the correlated bivariate normal, Grubbs-Pearson χ^2, Grubbs-Patnaik χ^2, Grubbs-Liu χ^2, and Rayleigh distribution as explained in section 3.2.1.

In the example given below, we plug in the results for the 50%-CEP as calculated by `getCEP()` in section 3.2.1 for r, and therefore expect a hit probability of 50%.

```
## for the Grubbs-Patnaik estimate
getHitProb(DFscar17, r=0.8414825, unit="in", doRob=FALSE,
           dstTarget=100, conversion="yd2in", type="GrubbsPatnaik")

[1] 0.5

## for the Rayleigh estimate
getHitProb(DFscar17, r=0.8290354, unit="in", doRob=FALSE,
           dstTarget=100, conversion="yd2in", type="Rayleigh")

[1] 0.4632
```

Another calculation gives the estimated fraction of shots within a circle with radius 1 MOA.

```
getHitProb(DFscar17, r=1, unit="MOA", doRob=FALSE,
           dstTarget=100, conversion="yd2in", type="CorrNormal")

[1] 0.6508
```

3.2.3 Extrapolating CEP and confidence ellipse to different distances

Function `getCEP()` returns the radius of the circular error probable (CEP) in absolute and angular size units, as does `getConfEll()` for the size of the confidence ellipse (section 3.2.1). Since angular size measures can be converted back to absolute size for arbitrary distances (3.7.1), it is possible to estimate the absolute size of the CEP and confidence ellipse for distances different than the one a group was actually shot at.

Given an observed group shot at 100 yd, one might, for example, calculate the radius of the circle at 300 m that is expected to contain 50% of the shots. This calculation is highly idealized as it makes the assumption that all influences on precision scale linearly with distance. Under most circumstances, this assumption is invalid. Generally, extrapolating beyond observed data can often be misleading. However, projecting CEP to slightly different distances might still give a sufficient approximation.

```
## 50% circular error probable for group shot at 100yd
CEP100yd <- getCEP(DFscar17, type=c("GrubbsPatnaik", "Rayleigh"),
                   CEPlevel=0.5, dstTarget=100, conversion="yd2in")

## CEP in absolute and angular size units
CEP100yd$CEP

$CEP0.5
     GrubbsPatnaik Rayleigh
unit        0.8415   0.8751
MOA         0.8036   0.8357
SMOA        0.8415   0.8751
mrad        0.2337   0.2431

## extract CEP in MOA
CEPmoa <- CEP100yd$CEP$CEP0.5["MOA", c("GrubbsPatnaik", "Rayleigh")]

## 50% CEP in inch for the same group extrapolated to 100m
fromMOA(CEPmoa, dst=100, conversion="m2in")

GrubbsPatnaik      Rayleigh
        55.22         57.42
```

Given a group shot at 100 yd, one may be interested in the expected fraction of shots within a circular region with radius $r = 1$ inch around the group center at the distance of 100 m (section

3.2.2). To this end, we first convert 1 inch at 100 m to MOA, and then supply the MOA value to `getHitProb()`.

```
## 1 inch at 100 m in MOA
MOA <- getMOA(1, dst=100, conversion="m2in")

getHitProb(DFscar17, r=MOA, unit="MOA", doRob=FALSE,
           dstTarget=100, conversion="yd2in", type="GrubbsPatnaik")

[1] 0.0005236
```

3.2.4 Literature related to CEP

The literature on the circular error probable (CEP) is extensive and diverse: Applications for CEP are found in areas such as target shooting, missile ballistics, error analysis for geopositional asessments, or positional accuracy of navigation and guidance systems like GPS. The statistical foundations in quadratic forms of normal variables are important for analyzing the power of inference tests. The Hoyt and Rayleigh distribution have applications in (wireless) signal processing, the Rice distribution is important for medical imaging techniques like MRI, the Maxwell-Boltzmann distribution comes from the physics of ideal gases.

The following list is not intended to be complete. Beware that the quality of the cited articles is not uniformly high. The relevant publications may be roughly categorized into different groups:

- Articles that develop a CEP estimator or the modification of one – e. g., RAND-234 (RAND Corporation, 1952), modified RAND-234 (Pesapane & Irvine, 1977), Grubbs (1964a), Rayleigh (Culpepper, 1978; Saxena & Singh, 2005; H. P. Singh, 1992), Ethridge (1983), Bell (1973), Nicholson, Braddock, and McDonald (1974), Siouris (1993), Krempasky (2003), Ignani (2010), correlated bivariate normal (DiDonato & Jarnagin, 1961a; Evans et al., 1985; Winkler & Bickert, 2012), and Bayes (Spall & Maryak, 1992).
- Some articles focus on the confidence intervals for CEP (DiDonato, 2007; Sathe, Joshi, & Nabar, 1991; Taub & Thomas, 1983b; Thomas, Crigler, Gemmill, & Taub, 1973b; Thomas & Crigler, 1974; Zhang & An, 2012).
- Articles, technical reports or Master's theses comparing the characteristics of CEP estimators in different scenarios (Blischke & Halpin, 1966; Elder, 1986; Kamat, 1962; McMillan & McMillan, 2008; Moranda, 1959, 1960; Nelson, 1988; Puhek, 1992; Tongue, 1993; Taub & Thomas, 1983a; Wang, Jia, Yang, & Wang, 2013; Wang, Yang, et al., 2013; Wang et al., 2014; Williams, 1997; Yakimenko, 2013; Carlson & Beer, 2021).
- Publications studying the correlated bivariate normal distribution with mean 0 re-written in polar coordinates radius and angle (Chew & Boyce, 1962; Greenwalt & Shultz, 1962; Harter, 1960; Hoover, 1984; Hoyt, 1947; Weingarten & DiDonato, 1961). The distribution of the radius is known as the Hoyt (1947) distribution. The closed form expression for its cumulative distribution function has only recently been identified as the symmetric difference between two first-order Marcum Q-functions (Marcum, 1950; Paris, 2009a, 2009b). The latter are special cases of the non-central χ^2-distribution (Nuttall, 1975). A

nearly correct closed-form solution for the 50% quantile is given by Krempasky (2003). Childs, Coffey, and Travis (1975) and Ignani (2010) provide polynomial approximations for the 50%, 90%, 95%, and 99% quantile. The statistical literature on coverage problems in the multivariate normal distribution is reviewed in Guenther and Terragno (1964). From the perspective of radio engineering, Beckmann (1962b, 1962a) works out the relationship between the Rayleigh, Rice and Hoyt distribution and their generalization.

- Cadwell (1964); DiDonato and Jarnagin (1961a, 1961b, 1962a, 1962b); Evans et al. (1985); D. C. Gilliland (1962); D. Gilliland and Hansen (1974); Moranda (1960) as well as Ager (2004); Bell (1973); Evans et al. (1985); Shultz (1963) develop methods to use the correlated bivariate normal distribution for CEP estimation when systematic accuracy bias must be taken into account. This requires integrating the distribution over a disc that is not centered on the true mean of the shot group but on the point of aim. This so-called *offset circle probability* is the probability of a quadratic form of a normal variable (Duchesne & Lafaye de Micheaux, 2010). The exact distribution of quadratic forms is a weighted average of non-central χ^2-distributions and difficult to calculate without numerical tools. Therefore, the Patnaik (1949) two-moment central χ^2-approximation or the Pearson (Imhof, 1961; Pearson, 1959) three-moment central χ^2-approximation are often used. Liu et al. (2009) proposed a four-moment non-central χ^2-approximation.
- A number of articles present algorithms for the efficient numerical calculation of the Hoyt cumulative distribution function (cdf), as well as for its inverse, the quantile function (DiDonato, 2004, 2007; Gillis, 1991; Govindarajulu, 1986; Pyati, 1993; Rogers, 1993; Shnidman, 1995). Algorithms to efficiently and precisely calculate the distribution of general quadratic forms of normal random variables include numerical integration Davies (1980); Farebrother (1984, 1990); Imhof (1961); Sheil and O'Muircheartaigh (1977) and the saddlepoint approximation (Kuonen, 1999). A comparison and implementation can be found in Duchesne and Lafaye de Micheaux (2010). Other comparisons include Bodenham and Adams (2016) and Chen and Lumley (2019).
- Eckler (1969); Eckler and Burr (1972) focus on the application of CEP in bombing tests.
- Peiliang et al. (2019); Zimmer, Park, and Mathew (2016); Spall and Maryak (1992); Spall (1997) analyze CEP for a mixture of bivariate distributions.
- The Spherical Error Probable (SEP) is developed in Childs et al. (1975); DiDonato (1988); Ignani (2010); Schulte (1968); N. Singh (1962, 1970); Siouris (1993); Thomas, Crigler, Gemmill, and Taub (1973a).
- Papp and Rožnijk (2022) use simulation and stochastic approximation.
- The GPS-related literature on accuracy measures in navigation includes Mertikas (1985) and Chin (1987).

3.3 Distributions for radial error

`shotGroups` implements several distributions for radial error in multivariate normal variables which apply to different situations. In general, the following functions are available:

- `d`⟨`Name`⟩`()`: The probability density function (pdf).
- `p`⟨`Name`⟩`()`: The cumulative distribution function (cdf, the integral over the pdf).
- `q`⟨`Name`⟩`()`: The quantile function (cdf^{-1}, the inverse of the cdf).

- `r⟨Name⟩()`: The function that generates random numbers from the given distribution.

The following distributions are available, illustrated in figure 1:

- Rayleigh: The radius around the true mean in a bivariate uncorrelated normal random variable with equal variances, re-written in polar coordinates (radius and angle), follows a Rayleigh distribution. Fully defined in closed form.
- Maxwell-Boltzmann: The radius around the true mean in a trivariate uncorrelated normal random variable with equal variances, re-written in polar coordinates (radius, azimuth, elevation), follows a Maxwell-Boltzmann distribution. Fully defined in closed form (using the cdf of the normal distribution).
- Rice: The radius around the origin in a bivariate uncorrelated normal random variable with equal variances and an offset mean, re-written in polar coordinates (radius and angle), follows a Rice distribution. The pdf, cdf, and inverse cdf are defined in closed form with the Marcum Q-function. Reduces to the Rayleigh distribution if the mean has no offset.
- Hoyt: The radius around the true mean in a bivariate correlated normal random variable with unequal variances, re-written in polar coordinates (radius and angle), follows a Hoyt distribution. The pdf and cdf are defined in closed form, numerical root finding is used to find cdf^{-1}. Reduces to the Rayleigh distribution if the correlation is 0 and the variances are equal.
- General case: The distribution of the radius around the origin in a bivariate correlated normal variable with unequal variances and an offset mean. The cdf of radial error is equal to the integral of the bivariate normal distribution over an offset disc. Not defined in closed form. Numerical integration provided by package `CompQuadForm` or a saddlepoint approximation (Kuonen, 1999) for quadratic forms of normal random variables is used to find the pdf and cdf (figure 2). Numerical root finding is used to find cdf^{-1}. Reduces to the Rice distribution if the correlation is 0 and the variances are equal. Reduces to the Hoyt distribution if the mean has no offset.

```
## probability of staying within 10cm of the point of aim
## Rayleigh distribution
pRayleigh(10, scale=5)

[1] 0.8647

## Rice distribution with offset x=1, y=1
pRice(10, nu=sqrt(2), sigma=5)

[1] 0.8538

## Hoyt distribution - unequal variances
sdX <- 8                              # standard deviation along x
sdY <- 4                              # standard deviation along y
hp  <- getHoytParam(c(sdX^2, sdY^2))  # convert to Hoyt parameters
pHoyt(10, qpar=hp$q, omega=hp$omega)
```

```
[1] 0.7332

## general case: unequal variances and offset x=1, y=1
sigma <- cbind(c(52, 20), c(20, 28))  # covariance matrix
pmvnEll(r=10, sigma=sigma, mu=c(1, 1), e=diag(2), x0=c(0, 0))

[1] 0.7251
```

The following examples show how the general correlated normal case encompasses the specialized distributions for radial error. The radial error in each case is 1.5. First, the case of equal variances and no offset mean.

```
## 1D - normal distribution with mean 0 for interval [-1.5, 1.5]
pnorm(1.5, mean=0, sd=2) - pnorm(-1.5, mean=0, sd=2)

[1] 0.5467

pmvnEll(1.5, sigma=4, mu=0, e=1, x0=0)

[1] 0.5467

## 2D - Rayleigh distribution
pRayleigh(1.5, scale=2)

[1] 0.2452

pmvnEll(1.5, sigma=diag(rep(4, 2)), mu=rep(0, 2), e=diag(2), x0=rep(0, 2))

[1] 0.2452

pmvnEll(1.5, sigma=diag(rep(4, 2)), mu=rep(0, 2), e=diag(2), x0=rep(0, 2),
        method_cdf="saddlepoint")

[1] 0.2456

## 3D - Maxwell-Boltzmann distribution
pMaxwell(1.5, sigma=2)

[1] 0.09504

pmvnEll(1.5, sigma=diag(rep(4, 3)), mu=rep(0, 3), e=diag(3), x0=rep(0, 3))

[1] 0.09504

pmvnEll(1.5, sigma=diag(rep(4, 3)), mu=rep(0, 3), e=diag(3), x0=rep(0, 3),
        method_cdf="saddlepoint")

[1] 0.09522
```

Next, the case of equal variances and an offset mean.

```
## 1D - normal distribution with mean 1 for interval [-1.5, 1.5]
pnorm(1.5, mean=1, sd=2) - pnorm(-1.5, mean=1, sd=2)

[1] 0.4931

pmvnEll(1.5, sigma=4, mu=1, e=1, x0=0)

[1] 0.4931

## 2D - Rice distribution
pRice(1.5, nu=1, sigma=2)

[1] 0.22

pmvnEll(1.5, sigma=diag(c(4, 4)), mu=c(1, 0), e=diag(2), x0=c(0, 0))

[1] 0.22

pmvnEll(1.5, sigma=diag(c(4, 4)), mu=c(1, 0), e=diag(2), x0=c(0, 0),
        method_cdf="saddlepoint")

[1] 0.2204
```

Next, the case of unequal variances and no offset mean.

```
## 2D - Hoyt distribution
sdX <- 4                              # standard deviation along x
sdY <- 2                              # standard deviation along y
hp  <- getHoytParam(c(sdX^2, sdY^2))  # convert to Hoyt parameters
pHoyt(1.5, qpar=hp$q, omega=hp$omega)

[1] 0.1291

pmvnEll(1.5, sigma=diag(c(sdX^2, sdY^2)), mu=c(0, 0), e=diag(2), x0=c(0, 0))

[1] 0.1291

pmvnEll(1.5, sigma=diag(c(sdX^2, sdY^2)), mu=c(0, 0), e=diag(2), x0=c(0, 0),
        method_cdf="saddlepoint")

[1] 0.132
```

3.4 Inference from range statistics and efficiency calculations

When it seems too costly to collect (x, y)-coordinates of all shots, range statistics like extreme spread, figure of merit, or the length of the bounding box diagonal are often measured instead (section 3.1). The statistical distribution of these measures depends on the number of shots per group n, and on the true spread of the distribution, but its exact form is unknown (Taylor & Grubbs, 1975). Nevertheless, it is possible to use range statistics for inference on the true spread of the distribution (Rayleigh parameter σ, section 3.2.1) when assuming a circular bivariate normal shot distribution with 0 mean.

Following the approach from Taylor and Grubbs (1975), a Monte Carlo simulation of circular bivariate normal shot groups with 0 mean and standard deviation 1 in both directions was carried out, see also Gammon (2017). The result is a lookup table for the distribution of extreme spread, figure of merit, bounding box diagonal, and Rayleigh σ. The table records the first four moments (mean, variance, skewness, kurtosis), the coefficient of variation, as well as the median and other major quantiles of the Monte Carlo distribution over 2 million repetitions in each scenario. One scenario was a combination of the n shots in each group, and the n_g groups over which individual range statistics were averaged. Values for n were 2, 3, ..., 49, 50, 55, ..., 95, 100. Values for n_g were 1, 2, ..., 9, 10. The lookup table with the simulation results is available as data frame `DFdistr`.

3.4.1 Distribution of range statistics

Based on the lookup table `DFdistr`, functions `pRangeStat()` and `qRangeStat()` give the approximate cumulative distribution function and the quantile function of range statistics within the parameter space covered by `DFdistr` (number of shots, number of groups, probabilities). For parameter combinations that were not simulated, interpolation between the supporting points is carried out. `rRangeStat()` generates random deviates.

```
# cumulative probability of extreme spread (ES) 4 and 5 with true
# Rayleigh sigma 1.5, and ES averaged over 3 groups with 5 shots each
(q45 <- pRangeStat(c(4, 5), sigma=1.5, n=5, nGroups=3, stat="ES"))

[1] 0.2057 0.7224

# quantiles for the returned probabilities, should be exactly 4 and 5
qRangeStat(q45, sigma=1.5, n=5, nGroups=3, stat="ES")

ES_Q206 ES_Q722
      4       5

# random deviates for bounding box diagonal
rRangeStat(5, sigma=2, nPerGroup=5, nGroups=3, stat="D")

[1] 7.044 6.192 5.758 5.957 8.685
```

3.4.2 Estimate Rayleigh σ from range statistics

Based on the lookup table from the Monte Carlo simulation, function `range2sigma()` estimates the Rayleigh σ parameter from measured range statistics – extreme spread, figure of merit, and the bounding box diagonal, assuming a circular bivariate normal shot distribution with 0 mean. This is possible because, conditional on n and n_g, range statistics directly scale with σ. Dividing the simulated mean range statistic for $\sigma = 1$ and the given n as well as n_g by the empirically measured range statistic therefore estimates σ. Confidence intervals can be inferred from the corresponding quantiles of the Monte Carlo distribution.

If σ should be estimated for a value of n that was not simulated (but is less than 100), a monotonic spline interpolation between the neighboring supporting points is used. If the desired coverage probability for the confidence interval corresponds to quantiles that were not calculated, the Monte Carlo distribution of squared extreme spread is approximated by a Patnaik χ^2 distribution as suggested by Taylor and Grubbs (1975).[7]

```
# get range statistics from DFscar17 data
es  <- getMaxPairDist(DFscar17)$d      # extreme spread
fom <- getBoundingBox(DFscar17)$FoM    # figure of merit
d   <- getBoundingBox(DFscar17)$diag   # bounding box diagonal

# estimate Rayleigh sigma from each statistic
range2sigma(c(es, fom, d), stat=c("ES", "FoM", "D"),
            n=nrow(DFscar17), nGroups=1, CIlevel=0.9,
            dstTarget=100, conversion="yd2in")

$sigma
     ES_3.048 FoM_2.445 D_3.463
unit   0.7996    0.7945  0.7832
MOA    0.7636    0.7587  0.7479
SMOA   0.7996    0.7945  0.7832
mrad   0.2221    0.2207  0.2176

$sigmaCI
$sigmaCI$ES
     sigma ( sigma )
unit  0.5967  1.1374
MOA   0.5698  1.0861
SMOA  0.5967  1.1374
mrad  0.1657  0.3159

$sigmaCI$FoM
     sigma ( sigma )
unit  0.6040  1.1126
MOA   0.5768  1.0624
SMOA  0.6040  1.1126
```

[7] Fort details, see http://ballistipedia.com/index.php?title=Range_Statistics

```
mrad  0.1678  0.3091

$sigmaCI$D
     sigma ( sigma )
unit  0.6781   1.293
MOA   0.6475   1.234
SMOA  0.6781   1.293
mrad  0.1883   0.359
```

Compare the results with estimating Rayleigh σ from using (x, y)-coordinates of all shots.

```
getRayParam(DFscar17, level=0.9)$sigma

  sigma sigCIlo sigCIup
 0.7432  0.5869  1.0290
```

Function `range2CEP()` directly estimates Rayleigh CEP from range statistics, by calling `getCEP()` based on the Rayleigh σ as estimated by `range2sigma()`.

3.4.3 Efficiency of group statistics

Assuming a circular bivariate normal shot distribution with 0 mean, the efficiency of using range statistics for inference on the Rayleigh σ parameter can be estimated when the distribution of the mean range statistic is assumed to be approximately normal – by appeal to the central limit theorem.

Given the number of shots per group n, it is then possible to determine the number of groups n_g that is required to achieve a desired coverage probability (level) for the confidence interval (CI) for σ with a desired CI width expressed as a fraction of the mean. Conversely, given n and the number of groups n_g over which the range statistic was averaged, we can estimate the CI width as a fraction of the mean for the given coverage probability. The CI width is a measure for the *statistical* precision of the estimator with more narrow CIs for the same level indicating better precision.[8]

Get the number of groups and total number of shots required to achieve a 90% CI with a CI width of 20% of the mean ($E = 10\%$ on either side), using 3, 5, or 10 shots per group and measuring extreme spread.

```
# ...Ceil gives the result when the number of groups is rounded
# up to the nearest integer
efficiency(n=c(3, 5, 10), CIlevel=0.9, CIwidth=0.2, stat="ES")

   n nGroupsReq nGroupsReqCeil nShotsReq nShotsReqCeil CIlevel CIwidth
1  3      37.09             38     111.3           114     0.9     0.2
2  5      19.68             20      98.4           100     0.9     0.2
3 10      10.24             11     102.4           110     0.9     0.2
```

[8] For details, noting that E in the references is half the CI width, see
http://ballistipedia.com/index.php?title=Range_Statistics
http://ballistipedia.com/images/3/32/Sitton_1990.pdf

Compare the result to directly estimating Rayleigh σ from (x, y)-coordinates.

```
efficiency(n=c(3, 5, 10), CIlevel=0.9, CIwidth=0.2, stat="Rayleigh")

   n nGroupsReq nGroupsReqCeil nShotsReq nShotsReqCeil CIlevel CIwidth
1  3     35.629             36    106.89           108     0.9     0.2
2  5     17.404             18     87.02            90     0.9     0.2
3 10      7.617              8     76.17            80     0.9     0.2
```

Given the simulated quantiles of the distribution for extreme spread, we can check that the result for $n = 10$ given above when assuming normality of mean extreme spread is approximately correct: The 5% quantile with $n_g = 10$ groups indeed is about 10% below the mean, and the 95% quantile with 10 groups is about 10% above the mean.

```
with(subset(DFdistr, (n == 10L) & (nGroups == 10L)),
     c(ES_Q050/ES_M, ES_Q950/ES_M))

[1] 0.9011 1.1033
```

Get the achievable 90% CI width as a fraction of the mean with 10 groups of 3, 5, 10 shots each using extreme spread.

```
efficiency(n=c(3, 5, 10), nGroups=10, CIlevel=0.9, stat="ES")

   n nGroups nShots CIlevel CIwidth
1  3      10     30     0.9 0.12182
2  5      10     50     0.9 0.08872
3 10      10    100     0.9 0.06400
```

3.5 Plotting scaled bullet holes on a target background

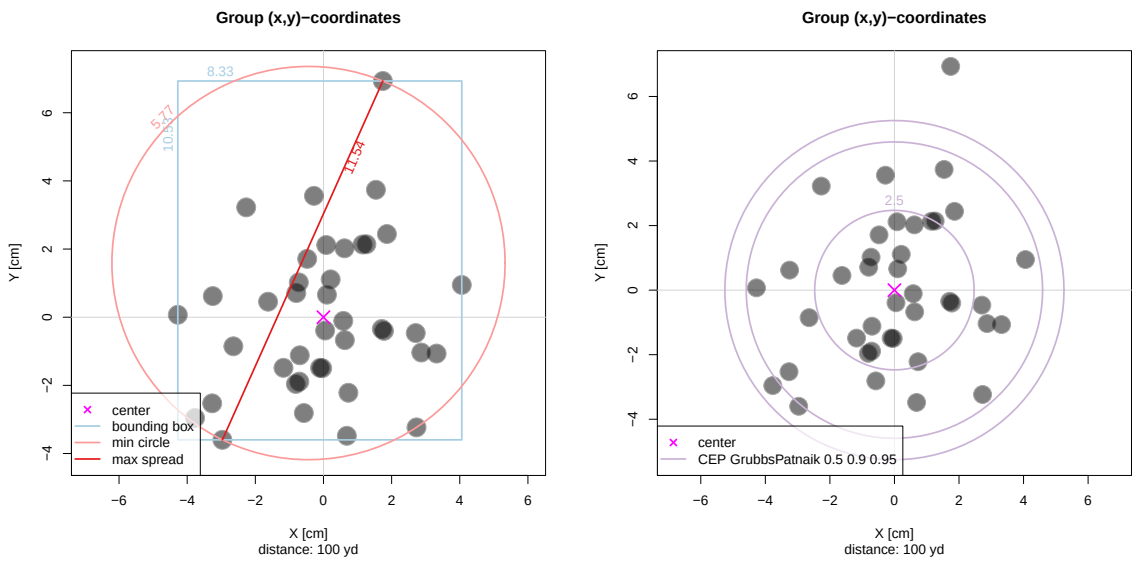
Function `drawGroup()` serves to illustrate a group of bullet holes by drawing the holes to scale on a target background, possibly adding the following features:

- The diagram can be drawn in original measurement units, in absolute size units m, cm, mm, yd, ft, in, or in angular measures deg, MOA, SMOA, rad, mrad, mil.
- A target background can be selected from a number of pre-defined circular target types from different shooting federations (ISSF, NRA, DSB, BDS, BDMP, DSU, see `help(targets)`). Targets can also be plotted just by themselves using `drawTarget()`.
- Precision indicators can be added to the plot individually:
  - (Minimum-area) bounding box with diagonal
  - Minimum enclosing circle
  - Maximum group spread
  - Circle with mean distance to group center

  - (Robust) confidence ellipse
  - Circular error probable CEP
- If a known target is supplied, the simulated ring value for each shot can be displayed (section 3.6)

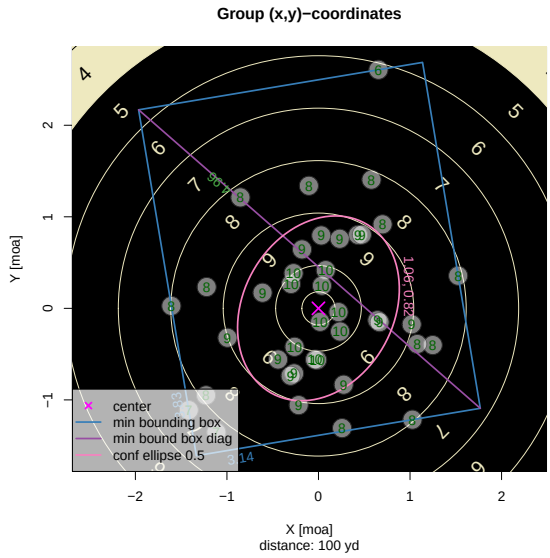
`drawGroup()` invisibly returns all the information that is shown in the diagram converted to the requested measurement unit. In the following example, the original measurement unit for (x, y)-coordinates was inch, the group is here drawn converted to cm. The second example shows how to plot a CEP estimator for multiple levels.

```
library(shotGroups, verbose=FALSE)       # load shotGroups package
dg1 <- drawGroup(DFcciHV, xyTopLeft=TRUE, bb=TRUE, minCirc=TRUE,
                 maxSpread=TRUE, scaled=TRUE, dstTarget=100,
                 conversion="yd2in", caliber=5.56, unit="cm", alpha=0.5,
                 target=NA)

## minimum enclosing circle parameters in cm
dg1$minCirc

$ctr
[1] -0.4395  1.5890

$rad
[1] 5.771

## show Grubbs CEP estimate for 50%, 90% and 95%
dg2 <- drawGroup(DFcciHV, xyTopLeft=TRUE, CEP="GrubbsPatnaik", scaled=TRUE,
                 level=c(0.5, 0.9, 0.95), dstTarget=100, conversion="yd2in",
                 caliber=5.56, unit="cm", alpha=0.5, target=NA)

## Grubbs CEP estimate for 50%, 90% and 95%
dg2$CEP

 CEP0.5  CEP0.9 CEP0.95
  2.471   4.592   5.254
```



Now draw the group with coordinates converted to MOA, add the minimum-area bounding box, 50%-confidence ellipse, use the ISSF 100 yd target, and show the ring value for each shot (section 3.6).

```
library(shotGroups, verbose=FALSE)       # load shotGroups package
dg3 <- drawGroup(DFcciHV, xyTopLeft=TRUE, bbMin=TRUE, bbDiag=TRUE,
                 confEll=TRUE, ringID=TRUE, level=0.5, scaled=TRUE,
                 dstTarget=100, conversion="yd2in", caliber=5.56, unit="MOA",
                 alpha=0.5, target="ISSF_100yd")

## simulated total ring count with maximum possible
dg3$ringCount

count   max
  351   400
```



3.6 Simulate ring count

Given the (x, y)-coordinates of a group, bullet diameter, and target type with definition of ring diameters, `simRingCount()` calculates a simulated ring count. This is an idealized calculation as it assumes that bullet holes exactly have the bullet diameter, and that rings exactly have the diameter given in the target definition. The count thus ignores the possibility of ragged bullet holes as well as the physical width of the ring markings. The simulated ring count therefore need not be equal to the calculated ring count from the corresponding physical target.

As an example, we simulate the ring count for the `DFscar17` data from shooting a .308 rifle (bullet diameter 7.62 mm) at 100 yd, using the ISSF target made for rifle shooting at 100 m.

```
library(shotGroups, verbose=FALSE)       # load shotGroups package
## simulated ring count and maximum possible with given number of shots
simRingCount(DFscar17, target="ISSF_100m", caliber=7.62, unit="in")

$count
[1] 71

$max
[1] 100

$rings
 [1] 7 6 7 7 7 7 7 7 8 8
Levels: 10 9 8 7 6 5 4 3 2 1 0
```

3.7 Conversion between absolute and angular size units

In addition to absolute length units, group size is often reported in terms of its angular diameter. Angles can be measured equivalently either in degree or in radian. If x is the angular

measurement in radian, and φ the angular measurement in degree for the same angle, then $\frac{x}{2\pi} = \frac{\varphi}{360}$ such that conversion between degree and radian is given by $x = \frac{2\pi}{360} \cdot \varphi$ and $\varphi = \frac{360}{2\pi} \cdot x$ (figure 3).

The angular size of an object with absolute size s is its angular diameter at a given distance d. This is the angle α subtended by the object with the line of sight centered on it (figure 4).

The `shotGroups` package includes functions `getMOA()` and `fromMOA()` to convert from absolute object size to the angular diameter in degree, MOA, SMOA, radian, mrad, NATO mil, and vice versa. The functions need the distance to target d, object sizes s and measurement units for d and s. The option `type` controls which angular measure is returned:

- `type="deg"`: Convert to/from angle in degree, where the circle is divided into 360 degrees.
- `type="MOA"`: Convert to/from MOA = minute of angle = arcmin. 1 MOA = 1/60 degree such that the circle is divided into $360 \cdot 60 = 21600$ MOA.
- `type="SMOA"`: Convert to/from SMOA = Shooter's MOA = Inches Per Hundred Yards IPHY. 1 inch at 100 yards = 1 SMOA.
- `type="rad"`: Convert to/from angle in radian, where 1 radian is 1 unit of arc length on the unit circle which has a circumference of 2π. The circle circumference is thus divided into 2π rad.
- `type="mrad"`: Convert to/from mrad = milliradian = 1/1000 radian. The circle circumference is divided into $2\pi \cdot 1000 \approx 6283.19$ mrad.
- `type="mil"`: Convert to/from NATO mil. 1 mil = $\frac{2\pi}{6400}$ radian – the circle circumference is divided into 6400 mils.

Function `getDistance()` returns the distance to an object given its absolute size and angular size in deg, MOA, SMOA, rad, mrad, or mil.

3.7.1 Calculating the angular diameter of an object

Figure 4 shows how the angle α subtended by an object of size s at distance d can be calculated from the right triangle with hypotenuse d and cathetus $s/2$: $\tan\left(\frac{\alpha}{2}\right) = \frac{s}{2} \cdot \frac{1}{d}$, therefore $\alpha = 2 \cdot \arctan\left(\frac{s}{2d}\right)$.

Assuming that the argument for $\tan(\cdot)$ and the result from $\arctan(\cdot)$ are in radian, and that distance to target d and object size s are measured in the same unit, this leads to the following formulas for calculating α in deg, MOA, SMOA as well as x in rad, mrad and NATO mil based on d and s:

- Angle α in degree: $\alpha = \frac{360}{2\pi} \cdot 2 \cdot \arctan\left(\frac{s}{2d}\right) = \frac{360}{\pi} \cdot \arctan\left(\frac{s}{2d}\right)$
- Angle α in MOA: $\alpha = 60 \cdot \frac{360}{\pi} \cdot \arctan\left(\frac{s}{2d}\right) = \frac{21600}{\pi} \cdot \arctan\left(\frac{s}{2d}\right)$
- Angle α in SMOA: By definition, size $s = 1$ inch at $d = 100$ yards (= 3600 inch) is 1 SMOA.

  Conversion factors to/from MOA are
  $\frac{21600}{\pi} \cdot \arctan\left(\frac{1}{7200}\right) \approx 0.95493$, and

  $\frac{\pi}{21600} \cdot \frac{1}{\arctan(1/7200)} \approx 1.04720$.

  $\alpha = \frac{\pi}{21600} \cdot \frac{1}{\arctan(1/7200)} \cdot \frac{21600}{\pi} \cdot \arctan\left(\frac{s}{2d}\right) = \frac{1}{\arctan(1/7200)} \cdot \arctan\left(\frac{s}{2d}\right)$

- Arc length x in rad: $x = 2 \cdot \arctan\left(\frac{s}{2d}\right)$.
- Arc length x in mrad: $x = 2000 \cdot \arctan\left(\frac{s}{2d}\right)$.
  Conversion factors to/from MOA are $\frac{21600}{2000\pi} \approx 3.43775$ and $\frac{2000\pi}{21600} \approx 0.29089$.
- Arc length x in NATO mil: $x = \frac{6400}{\pi} \cdot \arctan\left(\frac{s}{2d}\right)$.
  Conversion factors to/from MOA are $\frac{21600}{6400} = 3.375$ and $\frac{6400}{21600} \approx 0.2962963$.

```
## convert object sizes in cm to MOA, distance given in m
getMOA(c(1, 2, 10), dst=100, conversion="m2cm", type="MOA")

[1] 0.3438 0.6875 3.4377
```

Likewise, absolute object size s can be calculated from angular size and distance to target d:

- From angle α in degree: $s = 2 \cdot d \cdot \tan\left(\alpha \cdot \frac{\pi}{360}\right)$
- From angle α in MOA: $s = 2 \cdot d \cdot \tan\left(\alpha \cdot \frac{\pi}{60 \cdot 360}\right) = 2 \cdot d \cdot \tan\left(\alpha \cdot \frac{\pi}{21600}\right)$
- From angle α in SMOA: $s = \frac{21600}{\pi} \cdot \arctan\left(\frac{1}{7200}\right) \cdot 2 \cdot d \cdot \tan\left(\alpha \cdot \frac{\pi}{21600}\right)$
- From arc length x in rad: $s = 2 \cdot d \cdot \tan\left(x \cdot \frac{1}{2}\right)$
- From arc length x in mrad: $s = 2 \cdot d \cdot \tan\left(x \cdot \frac{1}{2000}\right)$
- From arc length x in NATO mil: $s = 2 \cdot d \cdot \tan\left(x \cdot \frac{\pi}{6400}\right)$

```
## convert from SMOA to object sizes in inch, distance in yard
fromMOA(c(0.5, 1, 2), dst=100, conversion="yd2in", type="SMOA")

[1] 0.5 1.0 2.0

## convert from object sizes in mm to mrad, distance in m
fromMOA(c(1, 10, 20), dst=100, conversion="m2mm", type="mrad")

[1]  100 1000 2000
```

Conversely, distance to target d can be calculated from absolute object size s and angular size:

- From angle α in degree: $d = \frac{s}{2} \cdot \frac{1}{\tan(\alpha \cdot \pi/360)}$
- From angle α in MOA: $d = \frac{s}{2} \cdot \frac{1}{\tan(\alpha \cdot \pi/21600)}$
- From angle α in SMOA: $d = \frac{s}{2} \cdot \frac{1}{\tan(\alpha \cdot \arctan(1/7200))}$
- From arc length x in rad: $d = \frac{s}{2} \cdot \frac{1}{\tan(x/2)}$
- From arc length x in mrad: $d = \frac{s}{2} \cdot \frac{1}{\tan(x/2000)}$
- From arc length x in NATO mil: $d = \frac{s}{2} \cdot \frac{1}{\tan(x \cdot \pi/6400)}$

```r
## get distance in yard from object size in inch and angular size in MOA
getDistance(2, angular=5, conversion="yd2in", type="MOA")

[1] 38.2

## get distance in m from object size in mm and angular size in mrad
getDistance(2, angular=0.5, conversion="m2mm", type="mrad")

[1] 4
```

3.7.2 Less accurate calculation of angular size

Sometimes, a slightly different angular size is reported as corresponding to absolute size s at distance d: This is the angle α' subtended by the object if it "sits" on the line of sight (figure 5). α' can be calculated from the right triangle with hypotenuse d and cathetus s: $\tan(\alpha') = \frac{s}{d}$, therefore $\alpha' = \arctan(\frac{s}{d})$.

If size s is small compared to distance d, the difference between the actual angular diameter α and approximate angular size α' is negligible, but it becomes noticeable once s gets bigger in relation to d (figure 6).

3.8 Included data sets

The `shotGroups` package includes a number of empirical data sets with shooting results:

- `DFlandy01`, `DFlandy02`, `DFlandy03`, `DFlandy04`: Groups and chronograph readings from shooting a .22LR rifle at 50 m or 50 yd (905 observations)[9]
- `DF300BLK`: One group of shooting a Noveske AR-15 rifle in 300BLK at 100 yd with factory ammunition (20 observations)[10]
- `DF300BLKhl`: Three groups of shooting a Noveske AR-15 rifle in 300BLK at 100 yd with handloaded ammunition (60 observations, see footnote 10)
- `DFcciHV`: Two groups of shooting a PWS T3 rifle in .22LR at 100 yd (40 observations, see footnote 10)
- `DFcm`: Several groups of shooting a 9x19mm pistol at 25 m (487 observations)
- `DFtalon`: Several groups of shooting a Talon SS air rifle at 10 m (180 observations)[11]
- `DFsavage`: Several groups of shooting a Savage 12 FT/R rifle in .308 Win at distances from 100 to 300 m (180 observations, see footnote 11)
- `DFscar17`: One group of shooting an FN SCAR 17 rifle in .308 Win at 100 yd (10 observations, see footnote 10)

[9] Thanks: Larry Landercasper & Albert Highe.
[10] Thanks: David Bookstaber http://ballistipedia.com/
[11] Thanks: Charles & Paul McMillan

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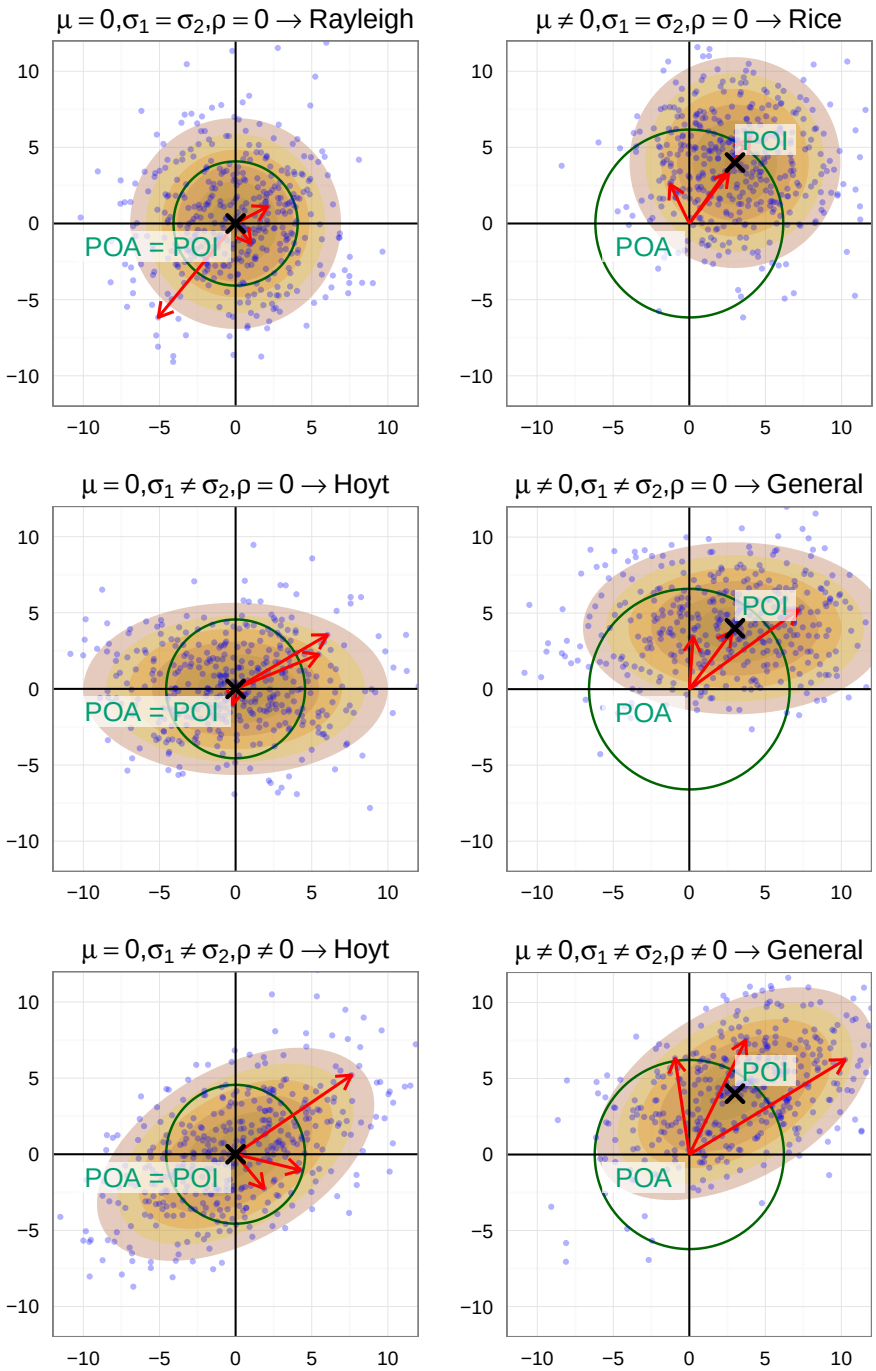

Figure 1: Distribution of radial error (red arrows) with 50% CEP (green circle) for several cases of bivariate normal shot distributions with point of aim (POA) and true mean (point of impact, POI). μ: center, σ: standard deviation, ρ: correlation

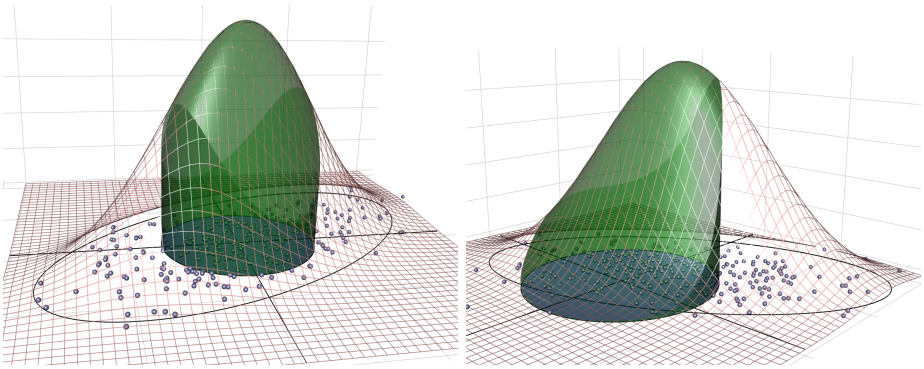

Figure 2: Bivariate normal distribution with 95% confidence ellipse, sample points, 50% CEP circle and corresponding volume under the normal surface. Left: Without systematic bias. Right: With systematic bias.

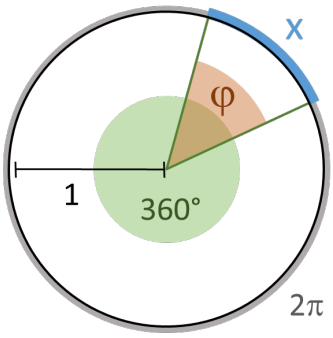

Figure 3: Angle φ (degree) with corresponding arc length x (radian) in the unit circle.

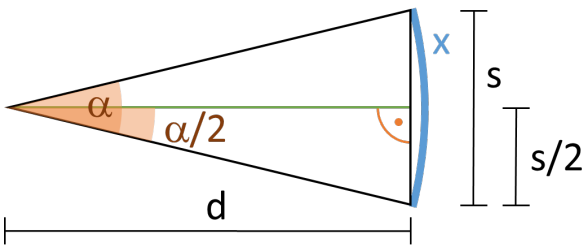

Figure 4: Angular diameter of object with absolute size s at distance to target d. Right triangle formed by d and object of size $s/2$. s corresponds to angle α (degree) and arc length x (radian).

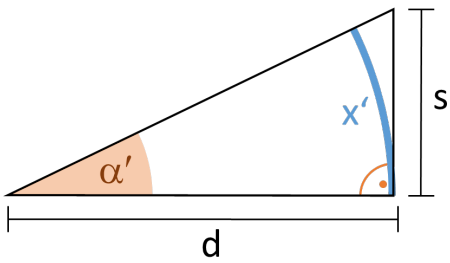

Figure 5: Object "sits" on line of sight: right triangle formed by distance to target d and object of size s. s corresponds to angle α' (degree) and arc length x' (radian).

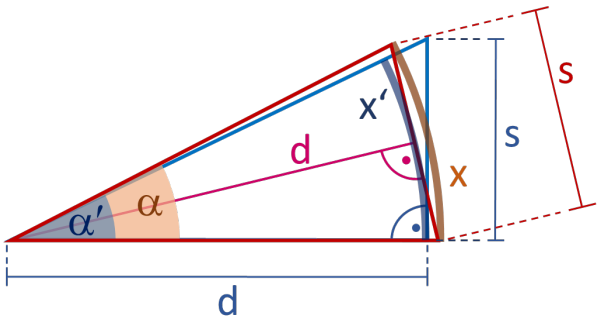

Figure 6: Comparison between actual angular diameter α (red) and the approximate angular size α' (blue) as well as between arc lengths x (red) and x' (blue) corresponding to s at distance d.